Bayesian Analysis of Duration Models: An Application to Chapter 11 Bankruptcy

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Abstract

We develop a Bayesian approach to estimating duration models and apply it to the default data of high yield bonds. The instantaneous probability of a firm completing Chapter 11 increases up to the twenty-first month in Chapter 11 then declines towards zero.

Keywords: Duration models; Bayesian inference; Laplace approximation; Chapter 11 bankruptcy; High yield bonds

JEL classification: C41; G33

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1. Introduction

The record number of bankruptcies in the United States during the late 1980s and early 1990s has created an opportunity for researchers to examine various aspects of Chapter 11 bankruptcy. This paper examines a model of the instantaneous probability (hazard rate) of a firm's emergence from Chapter 11 using Bayesian methods.

The motivation for our analysis comes from the empirical observation that the length of time firms spend in Chapter 11 bankruptcy varies widely (see Li (1998) for a survey), which leads to some interesting questions: Why do we observe such large variation in the time spent under formal reorganization? Can this variation in bankruptcy duration be explained by the financial/industrial characteristics of the distressed firms? Is it true that the longer a firm stays in Chapter 11, the more likely it exits from Chapter 11? Answers to questions such as these are of great interest to policy makers, practitioners, lending institutions, the claimants and owners of the bankrupt firm.

This paper uses a novel econometric model framework to examine the length of time that distressed original issuers of high yield bonds spend in Chapter 11 bankruptcy. Our work distinguishes itself from the existing literature in the following way. First, the duration model approach enables us to deal with sample firms that were still in their Chapter 11 proceedings at the end of the sample period (censored observations). Second, the Log-Logistic hazard formulation captures the institutional feature of Chapter 11 bankruptcy and accounts for some of the very lengthy Chapter 11 cases in the sample. Finally, in this paper we develop a Bayesian approach to the duration data analysis. The Bayesian methodology takes into account both the parameter uncertainty (by computing the posterior density) and the model specification uncertainty (by considering three plausible hazard formulations). And the Bayesian estimates possess finite sample properties which are important to our small data set.

The outline of our paper is as follows. In the next section, we discuss our Bayesian inference technique. In Section 3, we apply this to the default data of high yield bonds. Section 4 concludes.

2. Bayesian Inference

In this section we first discuss the three hazard formulations, then we deal with our Bayesian method.

2.1. Model Framework

The period from a firm filing Chapter 11 ("in" Chapter 11) until its emergence ("out" of Chapter 11) is the focus of our study. The hazard rate is defined as the probability of a firm exiting Chapter 11 bankruptcy at time t *conditional* on it having stayed there for t periods.

Due to the lack of theoretical guidance on the functional form of the hazard, we start with three most commonly adopted hazard formulations—the Exponential, the Weibull, and the Log-Logistic (Kiefer (1988), Lancaster (1990), p. 17, pp. 44-45). The hazard function of the Exponential distribution, conditional on the regressors, X, is

$$h(t, X) = \lambda, \tag{1}$$

where t is the length of time spent in Chapter 11,

$$\lambda = \exp(-X\beta), \quad \text{and} \quad X\beta = \beta_0 + x_1\beta_1 + \cdots + x_K\beta_K. \tag{2}$$

The hazard function of the Weibull distribution, conditional on the regressors, X, is

$$h(t, X) = \gamma\lambda^{\gamma}t^{\gamma-1}, \tag{3}$$

and the hazard function of the Log-Logistic distribution is

$$h(t, X) = \frac{\gamma\lambda^{\gamma}t^{\gamma-1}}{1 + (\lambda t)^{\gamma}}. \tag{4}$$

In Eqs. (1), (3) and (4), the scale parameter, λ, captures the effect of a given regressor on the hazard rate (the duration). A positive (negative) β_k in Eq. (2) implies that the corresponding regressor, x_k, has a negative (positive) effect on the hazard, hence a positive (negative) effect on the Chapter 11 duration. The shape parameter, γ, captures the time evolution of the hazard. For instance, in the Weibull case, $\gamma > 1$ ($\gamma < 1$) implies positive (negative) duration dependence for the hazard, i.e., the probability of a firm exiting Chapter 11 increases (decreases) as the Chapter 11 procedure increases in length; when $\gamma = 1$ the Weibull reduces to the Exponential with a constant hazard. In the Log-Logistic case, $\gamma \leq 1$ implies negative duration dependence for the hazard; $\gamma > 1$ implies the hazard is inverted U-shaped, i.e., the hazard rises with the duration from the period 0 to a maximum at the period $t_i^* = \frac{(\gamma-1)^{1/\gamma}}{\lambda_i}$ then it declines towards zero thereafter. The Log-Logistic distribution provides a more flexible hazard formulation than both the Weibull and the Exponential.

2.2. Bayesian Method

Under the prior independence assumption

$$f(\beta, \gamma) \sim f(\beta) \cdot f(\gamma), \tag{5}$$

we adopt a multivariate normal prior $MVN(\beta_0, \Psi_0^{-1})$ for the regression parameter β (which is directly related to the scale parameter $\lambda = \exp(-X\beta)$) and a Gamma prior $G(a_0, b_0)$ for the shape parameter γ. Our reason for adopting a Gamma prior for the shape parameter γ is due to the consideration that by model construction, γ is nonnegative. A Gamma-distributed random variable satisfies this constraint. Given $\gamma \sim G(a_0, b_0)$, the prior mean $E(\gamma) = a_0 b_0$, and the prior variance $Var(\gamma) = a_0 b_0^2$.

At the outset the question arises of which hazard formulation—the Exponential (M_1), the Weibull (M_2) or the Log-Logistic (M_3) should be adopted for our final estimation. We apply the Bayesian model comparison criterion—the Bayes factor (Kass and Raftery (1995)) to these three hazard functions. The Bayes factor BF_{jl} is defined as the ratio of the marginalized likelihood functions of the data under models M_j and M_l. Table 1 reports some Bayes factors and we examine the robustness of our model comparison results using four different prior specifications. We find that no matter how different the prior specifications are, we can decisively reject both the Exponential and the Weibull models given the sample data. Henceforth, our focus will be given to estimating the regression parameter β and the shape parameter γ using the Log-Logistic hazard formulation in Eq. (4).

[Insert Table 1 here]

Assuming independence across observations, the likelihood function of the observed time in Chapter 11, t, of the sample firms is

$$l(\beta, \gamma, t, X) \propto \prod_{d_i=1} f(t_i, X_i|\beta, \gamma) \prod_{d_i=0} S(t_i, X_i|\beta, \gamma), \tag{6}$$

where $d_i = 1$ if the firm's Chapter 11 procedure is complete, zero otherwise. The first component of the above likelihood, $f(t_i, X_i|\beta, \gamma)$, accounts for the probability contribution from the firms that have emerged from Chapter 11 bankruptcy. While the second component of the likelihood function, $S(t_i, X_i|\beta, \gamma)$, accounts for the probability contribution from the firms that were still in their Chapter 11 proceedings at the end of the sample period. Substituting the Log-Logistic density function and survivor function to $f(\cdot)$ and $S(\cdot)$ respectively, we obtain the likelihood function of the sample firms as

$$l(\beta, \gamma, t, X) \propto \prod_{d_i=1} \frac{\gamma \lambda_i^{\gamma} t_i^{\gamma-1}}{1 + (\lambda_i t_i)^{\gamma}} \prod_{i} \frac{1}{1 + (\lambda_i t_i)^{\gamma}}, \tag{7}$$

where $\lambda_i = \exp(-X_i \beta)$.

Combining the prior and the likelihood function in Eq. (7), we get the posterior probability density function

$$f(\beta, \gamma|t, X) \propto \exp\left[-\frac{(\beta - \beta_0)' \Psi_0 (\beta - \beta_0)}{2}\right] \gamma^{a_0 - 1} \exp(-\gamma b_0^{-1}) \prod_{d_i=1} \frac{\gamma \lambda_i^{\gamma} t_i^{\gamma-1}}{1 + (\lambda_i t_i)^{\gamma}} \prod_{i} \frac{1}{1 + (\lambda_i t_i)^{\gamma}}. \tag{8}$$

There is no analytical formula for obtaining the posterior estimates of $\theta = (\beta', \gamma)'$. In this paper, we adopt the Tierney-Kadane Laplace approximation to compute all the posterior moments (Tierney, Kass and Kadane (1989), and references therein).

The Tierney-Kadane approximation is based on Laplace's method for asymptotic expansion of integrals. Assuming p is a smooth function of an m-dimensional parameter θ with $-p$ having a maximum at $\widehat{\theta}$, Laplace's method approximates an integral of the form

$$I = \int_{\Theta} \exp[-np(\theta)]d\theta \tag{9}$$

by

$$\widehat{I} = (\frac{2\pi}{n})^{1/2}(\det\Sigma)^{1/2}\exp[-np(\widehat{\theta})] \tag{10}$$

where Θ is the domain of θ, $\widehat{\theta}$ maximizes $p(\theta)$, Σ^{-1} is the corresponding Hessian evaluated at $\widehat{\theta}$, and n is the sample size. In a nutshell, the Tierney-Kadane Laplace method transforms an integration problem in Eq. (9) to a straightforward optimization problem of $p(\theta)$ in Eq. (10).

Given the posterior density $f(\theta|y)$ on Θ is unimodal and proportional to $\exp[-np(\theta)]$, $g(\theta)$ is a function of interest. The general task of Bayesian inference is to obtain the expected value of $g(\theta)$ under the posterior density in Eq. (8)

$$E[g(\theta)] = \int g(\theta)f(\theta|y)d\theta = \frac{\int g(\theta)f(\theta)l(\theta,y)d\theta}{\int f(\theta)l(\theta,y)d\theta}, \tag{11}$$

which can be approximated by applying Laplace's method to both the numerator and denominator of Eq. (11) at the same time. In particular, let $-p = [\ln f(\theta) + \ln l(\theta,y)]/n, -p^* = [\ln f(\theta) + \ln l(\theta,y) + \ln g(\theta)]/n$, Eq. (11) can be rewritten as

$$E[g(\theta)] = \frac{\int \exp(-np^*)d\theta}{\int \exp(-np)d\theta}, \tag{12}$$

applying the Tierney-Kadane Laplace approximation,

$$\widehat{E}[g(\theta)] = \frac{(\det\Sigma^*)^{1/2}\exp[-np^*(\widehat{\theta^*})]}{(\det\Sigma)^{1/2}\exp[-np(\widehat{\theta})]}, \tag{13}$$

where $\widehat{\theta}$ and $\widehat{\theta^*}$ are the maxima; Σ and Σ^*, minus the inverse Hessians of $-p$ and $-p^*$, respectively. An approximation for the posterior variance of $g(\theta)$ can also be obtained by setting

$$\widehat{V}[g(\theta)] = \widehat{E}[g^2(\theta)] - \widehat{E}[g(\theta)]^2. \tag{14}$$

When $g(\theta)$ is nonpositive, the Tierney-Kadane method still applies by adding a large constant to $g(\theta)$ and subtracting it later (Tierney, Kass and Kadane (1989)).

The computational requirement of the Tierney-Kadane approximation is rather minimal: we only need to evaluate first and second derivatives and maximize the two integrals $-p, -p^*$, which are simply modified likelihood functions. Many existing optimization packages can perform the task.

3. Some Empirical Results

In this section we present our results obtained for a sample of distressed original issue high yield bonds between 1980-1991 in which the issuers filed Chapter 11 (see Li (1998) for detailed description of the data).

[Insert Table 2 here]

Table 2 gives the posterior parameter estimates using the Log-Logistic hazard formulation. We find a negative effect of a prepackaged Chapter 11 (PREPACK) on the Chapter 11 duration which is consistent with previous empirical observations (Tashjian et al. (1996)). Prepacks typically spend less time in Chapter 11 because the bankruptcy petition and the reorganization plan are filed together in prepackaged Chapter 11 cases. The negative effect of the PRECH11 variable confirms the trade-off between the length of Chapter 11 and the length of pre-Chapter 11 negotiation (Gilson et al. (1990), Tashjian et al. (1996)). The longer a firm spends in out-of-court negotiation prior to a Chapter 11 filing, the sooner it emerges from Chapter 11.

We also find that the presence of lawsuits (DISPUTE) extends a firm's stay in Chapter 11. The negative coefficient on the EBITDA/SALES variable is consistent with Jensen's (1991) viewpoint that firm value matters in resolving financial distress. It is relatively easier and faster for profitable firms to exit from Chapter 11. The TL variable measures the firm size prior to distress. Larger firms tend to stay longer in Chapter 11.

Firms that went into default in the 1990s and subsequently filed Chapter 11 spent less time in Chapter 11 than firms went into default prior to 1990 (POST90). We suspect that the courts and bankruptcy professionals became more experienced in dealing with large bankruptcies such as those in our sample at the end of this bankruptcy wave (early 1990s).

Finally, the posterior mean for the shape parameter γ is 3.3919. Under the Log-Logistic hazard model, $\gamma > 1$ implies an inverted U-shaped hazard. That is, the representative firm has an increasing hazard until the twenty-first month in Chapter 11 (t_i^*). Afterwards, the instantaneous probability of its exiting from Chapter 11 declines towards zero.

The finding of an inverted U-shaped hazard rate is consistent with the institutional feature of Chapter 11 (Giammarino (1989), White (1989)). After a Chapter 11 filing, there is a 120 day exclusive period given to management of the bankrupt firm followed by 60 days to get its reorganization plan approved if a plan is submitted, hence for the first 6 months or longer, the probability of exiting from Chapter 11 is low (yet not zero due to some quick exits by the prepacks in our sample). Only after the reorganization plan is formulated, is there a sharp increase in the hazard rate. The institutional setup of Chapter 11 helps explain the rising hazard portion (with respect to the time spent in Chapter 11) of the inverted U-shape. However as time goes on, it becomes harder and harder for the distressed firm to sell assets, or continuously obtain post-petition financing in order to get out of Chapter 11. The instantaneous probability of exiting Chapter 11 starts to decrease towards zero after the turning point t_i^*, which explains the declining hazard portion of the inverted U-shape. That is, the longer a firm stays in Chapter 11 (beyond the period t_i^*), the less likely for it to exit from Chapter 11. This declining portion of the hazard accounts for some of the very lengthy Chapter 11 cases in our sample, such as ANAC Merger (47 months), LTV Corp. (83 months), etc.

4. Concluding Remarks

In this paper, we have proposed a Bayesian approach to examining the length of time distressed high yield debt issuing companies spend in Chapter 11 bankruptcy. Through a model of the instantaneous probability (hazard rate) of a firm's emergence from Chapter 11, we find that the length of Chapter 11 bankruptcy is significantly affected by a firm's choice of the prepackaged

Chapter 11, the time it spends in pre-Chapter 11 negotiation, the interruption of legal disputes, its gross profit margin, size and the changing bankruptcy environment in the 1990s. We also conclude that for a representative sample firm, the time it spends in Chapter 11 increases its instantaneous probability of completing Chapter 11 up to its twenty-first month in Chapter 11. After that, this instantaneous probability of exiting Chapter 11 declines towards zero.

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Table 1
Results from Model Comparison

Panel A. Prior Specification

	Prior 1	Prior 2	Prior 3	Prior 4
β_0	0_q	0_q	0_q	0_q
Ψ_0^{-1}	$10{*}I_q$	$100{*}I_q$	$10{*}I_q$	$10{*}I_q$
a_0	4	4	10	2
b_0	4	4	10	10

Panel B. Model Comparison

BF_{31}	6.37e+10	9.28e+10	3.73e+14	7.26e+9
BF_{32}	265.97	305.70	6.21e+3	133.87

Each column corresponds to a different set of prior specifications on the regression parameter $\beta \sim MVN(\beta_0, \Psi_0^{-1})$ and the shape parameter $\gamma \sim G(a_0, b_0)$. $q = 1 + K$ is the dimension of $\beta, 0_q$ the vector of zeros, I_q the identity matrix.

BF_{31} is the Bayes factor comparing the Log-Logistic hazard model (M_3) with the Exponential hazard model (M_1). BF_{32} is the Bayes factor comparing the Log-Logistic hazard model (M_3) with the Weibull hazard model (M_2).

Table 2
Results from the Log-Logistic Hazard Model (under Prior 1)

Variable	Coefficient	s.e.
PREPACK	-1.3467	0.1286
PRECH11	-0.0193	0.0071
COMPLEXITY	0.0977	0.0973
HLT	-0.1414	0.0955
HY/TL	-0.2532	0.2659
DISPUTE	0.4232	0.1528
EBITDA/SALES	-0.5047	0.2988
TL	0.0879	0.0441
POST90	-0.3333	0.1055
IEBITDA/SALES	0.6393	0.5710
TERMPREM	-0.0146	0.0383
Constant	3.3106	0.3215
γ	3.3919	0.5267

PREPACK is one for firms filed for a prepackaged Chapter 11 bankruptcy, zero otherwise.
PRECH11 measures the length of time a firm spends in out-of-court negotiation prior to its Chapter 11 filing.
COMPLEXITY is one for firms with more than one layer of subordination among their original issue high yield debt, zero otherwise.
HLT is one for firms had a highly leveraged transaction, zero otherwise.
HY/TL is a firm's outstanding original issue high yield debt divided by its total liabilities.
DISPUTE is one for firms involved in either underfunded pensions, environmental liabilities, subordination lawsuits among creditors, or major torts, zero otherwise.
EBITDA is earnings before interest, taxes, depreciation and amortization.
EBITDA/SALES is a firm's EBITDA divided by its sales.
TL is total liabilites in book value.
POST90 is one for defaults that took place after 1989, zero otherwise.
IEBITDA/SALES is the default firm's industry average EBITDA/SALES.
TERMPREM is the difference between the 30-year U.S. government bond interest rate and the 3-month Treasury bill rate.
Prior 1 refers to a set of prior specification on the regression parameter $\beta \sim MVN(\beta_0, \Psi_0^{-1})$ and the shape parameter $\gamma \sim G(a_0, b_0)$, with $\beta_0 = 0_q$, $\Psi_0^{-1} = 10 * I_q$, $a_0 = 4$, $b_0 = 4$.
$q = 1 + K$ is the dimension of $\beta, 0_q$ the vector of zeros, I_q the identity matrix.
s.e. gives the corresponding posterior standard error.